ALGEBRA 2

POLYNOMIAL FUNCTIONS

BEHAVIORAL OBJECTIVES

I. Define

  A. A polynomial function
  B. The zero of a function

II. State the Factor Theorem

III. Given a polynomial function determine

  A. The degree of the polynomial
  B. The number of terms of the polynomial
  C. The value of the function for a given domain element
  D. Its prime factors
  E. The domain elements x such that $P(x) > 0$
  F. The domain elements x such that $P(x) < 0$
  G. The zeros of the function
  H. The domain of the function
  I. The range of the function
  J. The graph of the function

SECTION I — THE POLYNOMIAL FUNCTION

MARSH FUNDAMENTAL THEOREM OF ALGEBRA 2:

Hypothesis: Algebra 2 students "LOVE" functions and have been known to perform well with them.

Conclusion: Algebra 2 students will LOVE Polynomial Functions and will show their genius at work in the exam.

Definition: A term in a mathematical expression, is a number or a letter standing alone, or several variables and/or constants joined only by signs of multiplication or division.

Examples: 3, a, $2x$, $3x^3y$, $2x/y$

Definition: A polynomial is an expression consisting of one or more terms where the terms are numbers or variables, or products of numbers and variables.

Some polynomials have special names; i.e.:

A. A polynomial with one term: $2xy$ (monomial)

B. A polynomial with two terms: $2x + 4y$ (binomial)

C. A polynomial with three terms: $2x^2 + 4x + 7$ (trinomial)

Definition: The degree of a term is the sum of the exponents of the variables in the term.

Examples: Term: $10x^2y$ Degree: 3

Term: $2^2x^4y^2z$ Degree: 7

A polynomial function of x consists of multiples of powers of x, whole-number powers giving terms that are usually listed in descending powers of x. Poly-comes from a Greek word meaning "many", and a polynomial is literally an expression of "many terms." At times this may be an exaggeration. Actually, a monomial, a binomial, and a trinomial sometimes fit into the classification of polynomials.

Definition: A polynomial funtion of degree n, is a function of the form:

$$f(x) = a_nx^n + a_{n-1}x^{n-1} + x_{n-2}x^{n-2} + \ldots + a_1x + a_0$$

where: $a_n, a_{n-1}, a_{n-2}, \ldots a_1$, and a_0 are constants and $a_n \neq 0$

n is a positive integer.

Definition: The degree of a polynomial function is just the highest exponent appearing.

Example: $f(x) = 6x^7 + 5x^5 + 9x^3 + 3x^2 + 4x + 8$

The degree of the polynomial is 7.

$a_7 = 6;\ a_6 = 0;\ a_5 = 5;\ a_4 = 0;\ a_3 = 9;\ a_2 = 3;\ a_1 = 4;$ and $a_0 = 8$

Note: The lowest possible degree of a polynomial is 0. Example: $f(x) = 3$. *

The polynomial function with degree 0 is a constant function.

A polynomial function with degree 1 is a linear function:

$$f(x) = a_1x + a_0$$

A polynomial function with degree 2 is a quadratic function:

$$f(x) = a_2x^2 + a_1x + a_0$$

The domain of a polynomial function is all real numbers.

We shall show later that the range of a polynomial function is

1) All reals if the degree of the polynomial is odd.
2) Not all reals if the degree of the polynomial is even.

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*The function $f(x) = 0$ is a constant function. It is actually a polynomial and called a polynomial with no degree, since it does not really fit the definition.

$f(x) = a_0,\ a_0 \neq 0$ IS A POLYNOMIAL FUNCTION OF DEGREE ZERO.

EXERCISE 1

A. Determine the degree of each of the following expressions:

1. $2x^2y$ 2. $24x^3y^3$ 3. $14x^{30}$ 4. $x^2y^2z^2$

B. Determine the degree of each of the following polynomial functions:

1. $P(x) = x^3 + 7x^2 + 10$ 2. $P(x) = 2 + 3x + 4x^2 + 8x^4$

3. $P(x) = 7x^5 + 2x + 1$ 4. $P(x) = 9$ 5. $P(x) = 0$

C. Write the general polynomial function of degree:

1. 4 2. 6 3. 1 4. 3 5. 2

D. Which polynomial functions in C have a range of all reals?

SECTION II EVALUATION OF P(x) FOR A GIVEN x

The domain of a polynomial function is the set of all real numbers. It is rather a simple task to evaluate a polynomial function for a given domain element... although it is sometimes tedious. Study the following examples:

Example 1: $P(x) = x^3 - 9x^2 + 24x - 20$

Evaluate $P(2)$

$$P(2) = 2^3 - 9(2)^2 + 24(2) - 20$$
$$= 8 - 36 + 48 - 20$$
$$= 0$$

Example 2: $P(x) = x^4 + x^3 - x^2 + 9$

Evaluate $P(\sqrt{2})$

$$P(\sqrt{2}) = (\sqrt{2})^4 + (\sqrt{2})^3 - (\sqrt{2})^2 + 9$$
$$= 4 + 2\sqrt{2} - 2 + 9$$
$$= 11 + 2\sqrt{2}$$

EXERCISE 2 Evaluate each of the following:

A. $P(x) = 2x^3 - 3x^2 + 2x - 1$; 1. $P(2) =$ 2. $P(0) =$

B. $P(x) = 5x^3 - 3x^2 + 2x - 5$; 1. $P(2) =$ 2. $P(-1) =$

C. $P(x) = 3x^4 - 6x^3 - 2x^2 + x - 8$; 1. $P(3) =$ 2. $P(.1) =$

D. $P(x) = x^5 - 2x^3 + 4x - 2$ 1. $P(-2) =$ 2. $P(0) =$

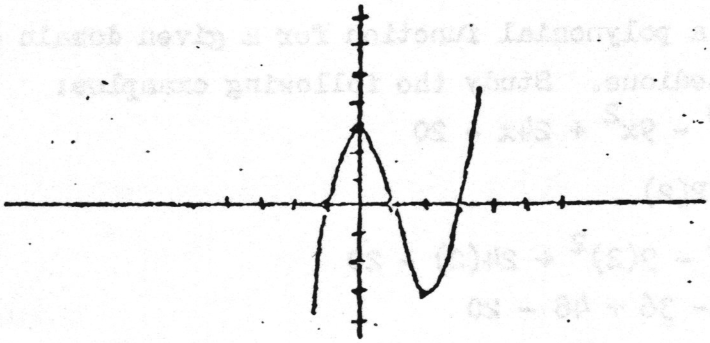
SECTION III THE FACTOR THEOREM

Consider the polynomial function $P(x) = x^3 - 3x^2 - x + 3$

This particular function is relatively easy to graph. First factor the polynomial.

$$\begin{aligned} P(x) &= x^3 - 3x^2 - x + 3 \\ &= x^2(x - 3) - (x - 3) \\ &= (x - 3)(x^2 - 1) \\ P(x) &= (x - 3)(x + 1)(x - 1) \end{aligned}$$

Once the polynomial is factored it yeilds itself to easy point location. Notice, when $x = 3, 1,$ or -1, the value of $P(x)$ is 0. Thus we have three ordered pairs of the graph: $(3,0)$, $(1,0)$, and $(-1,0)$. By letting x = numbers such as -2, 3, 2, and 4, additional points on the curve may be obtained.



-1, 1, and 3, for the above polynomial are called <u>zeros</u> of the polynomial.

Definition: A zero of a function f, is a number c in the domain of f such that $f(c) = 0$.

Not all polynomials cooperate as does the example, and factor so readily. However, we can scheme a bit and get around some situations.

Consider the polynomial: $P(x) = x^4 + a_3x^3 + a_2x^2 + a_1x + a_0$. (1)

Suppose there is a binomial factor $x - c$.

Then: $P(x) = (x - c)(x^3 + \ldots + a_0/c)$ (2)

Notice in (2) $P(c) = 0$. Consequently also in (1), $P(c) = 0$.
To achieve integral coefficients, notice also that c must divide a_0.

above hanky-panky is an attempt to somewhat justify the Factor Theorem which is gallantly displayed on the next page.

The Factor Theorem: For a polynomial function, $P(x)$, $P(x)$ is divisible by $(x - c)$ if and only if $P(c)$ equals zero.

Consider the polynomial $P(x) = x^3 - 9x^2 + 24x - 20$

The factors of $P(x)$ are not immediately apparent. Hence, we look for a number c such that $P(c) = 0$. Recall also, c should be a factor of 20.

The numbers to try are ± 1, ±2, ±4, ± 5, ± 10, and ± 20.

Immediately, in this case, we reject the negatives because the value of the polynomial would be negative and never zero.

So crank:...$P(1) = 1 - 9 + 24 - 20 = 4$ Reject

$P(2) = 8 - 36 + 48 - 20 = 0$ What is this? Victory already!

Now divide:

$$
\begin{array}{rl}
 & x^2 - 7x + 10 \\
x - 2 \,\big) & x^3 - 9x^2 + 24x - 20 \\
 & \underline{x^3 - 2x^2} \\
 & \quad - 7x^2 + 24x \\
 & \quad \underline{- 7x^2 + 14x} \\
 & \qquad\qquad 10x - 20 \\
 & \qquad\qquad \underline{10x - 20}
\end{array}
$$

Rip up the quotient: Fortunately, the quotient easily factors to $(x - 5)(x - 2)$

Hence, $P(x) = (x-2)^2(x-5)$. The zeros of the function are 2 and 5.

EXAMPLE 2: $P(x) = x^3 - 10x^2 + 29x - 24$

By checking the divisors of 24, notice that $P(3) = 0$.

Then divide $P(x)$ by $x - 3$. The quotient is $x^2 - 7x + 8$.

The quotient is not easily factored, but the quadratic formula can be used and we determine that the zeros are:

$$\frac{7 + \sqrt{17}}{2} \quad \text{and} \quad \frac{7 - \sqrt{17}}{2}$$

Hence, the zeros of $P(x)$ are: $\left\{3, \frac{7 + \sqrt{17}}{2}, \frac{7 - \sqrt{17}}{2}\right\}$

NOTE: If, when using the quadratic formula, the roots are imaginary, do not list those imaginary numbers as zeros of the function. We are working only over the domain of real numbers.

EXERCISE 3 Find the zeros for each of the following polynomials

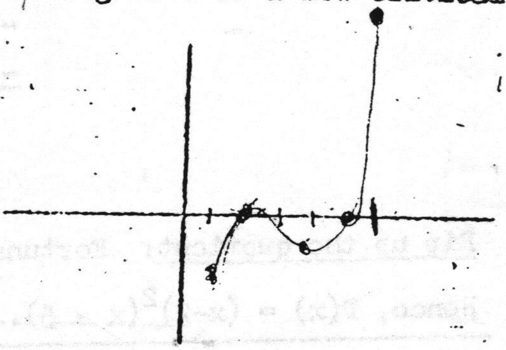
1. $P(x) = x^3 + 10x^2 + 29x + 20$
2. $P(x) = x^3 - 12x^2 + 29x - 18$
3. $P(x) = x^3 - 8x^2 + 21x - 18$
4. $P(x) = x^3 - 6x^2 + 11x - 6$
5. $P(x) = x^3 - 27x + 10$
6. $P(x) = x^4 - 10x^3 + 35x^2 - 50x + 24$
7. $P(x) = x^3 - 4x^2 + x + 6$
8. $P(x) = x^3 - 6x^2 + 11x - 6$
9. $P(x) = x^3 + 6x^2 + 11x + 6$
10. $P(x) = x^3 + 7x^2 + 7x - 15$
11. $P(x) = x^4 - 16$
12. $P(x) = x^4 - 8x$

SECTION IV GRAPHING POLYNOMIALS

Polynomial functions graph as smooth, continuous curves. A computer can whiz out millions of values for the y coordinates if given millions of values for x in just a matter of seconds. It takes the man on the street a little longer. But, we can do a good job of approximating the curve by simply taking care of a few critical points.

Example 1: $P(x) = x^3 - 9x^2 + 24x - 20$

$P(x) = (x - 2)^2 (x - 5)$



The zeros are 2 and 5. Plot them.

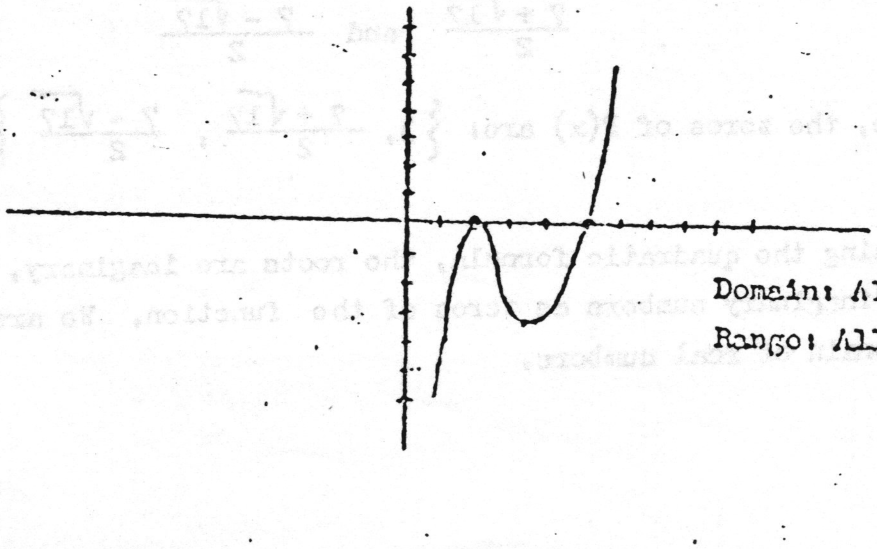
Now pick an x value to the right of 5; i.e.: 6. $P(6) = 16$

Pick a point midway between 2 and 5; i.e.: $3\frac{1}{2}$. $P(3\frac{1}{2}) = \frac{-27}{8}$

Pick a point to the left of 2; i.e.: 1; $P(1) = -4$

Plot the ordered pairs: $(6,16)$, $(3\frac{1}{2}, -\frac{27}{8})$; $(1,-4)$

Connect all points with a smooth curve.



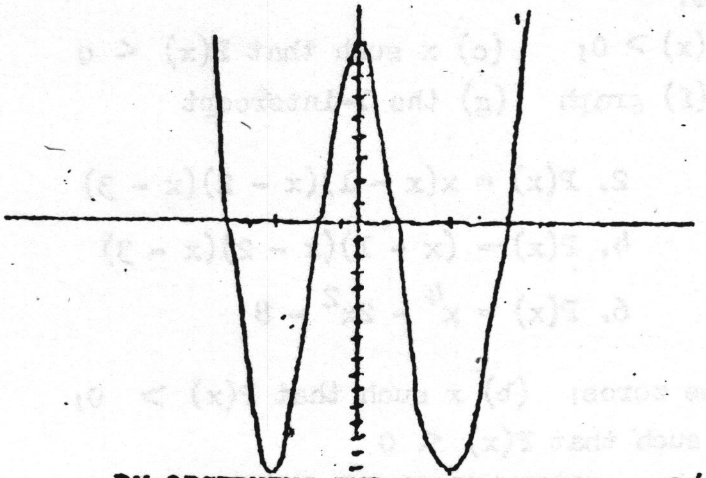
Domain: All reals

Range: All realls

EXAMPLE 2: $P(x) = x^4 - 10x^2 + 9$

$= (x^2 - 9)(x^2 - 1)$

$P(x) = (x + 3)(x - 3)(x + 1)(x - 1)$



$P(4) = 7 \cdot 15 = 105$

$P(-4) = 105$

$P(\pm 2) = -5 \cdot 3 = -15$

$P(0) = -9 \cdot -1 = 9$

BY OBSERVING THE GRAPH NOTICE: $P(x) > 0$ for $x < -3$ or $-1 < x < 1$ or $x > 3$

$P(x) < 0$ for $-3 < x < -1$ or $1 < x < 3$

Domain: All reals.

Range: $\{y: y \geq -15\}$ (Actually, it takes calculus to really determine the the range of this function. -15 will do for now.)

EXAMPLE 3: Determine the zeros of $P(x)$ and the values of x for which (a) $P(x) > 0$; and (b) $P(x) < 0$, Do this without graphing the function.

$P(x) = x(x + 4)(x - 3)(x + 1)$

(1) From observing the factors, the zeros are: 0, -4, 3, and -1

(2) Using a schematic sign graph:

x	- - - - - - - - - - - 0 + + + + + + + + + + + +	0
$x + 4$	- - - - - - - 0 + + + + + + + + + + + + + + + + +	-4
$x - 3$	- - - - - - - - - - - - - - - - - 0 + + + + + + +	3
$x + 1$	- - - - - - - - - - - 0 + + + + + + + + + + + + +	-1
Product:	+ + + + 0 - - - 0 + + 0 - - - - 0 + + + + + +	-4 -1 0 3

$P(x) > 0$ for $x < -4$ or $-1 < x < 0$ or $x > 3$

$P(x) < 0$ for $-4 < x < -1$ or $0 < x < 3$

EXERCISE 4

I. For each of the following determine:

(a) the zeros; (b) x such that $P(x) > 0$; (c) x such that $P(x) < 0$

(d) domain; (e) range; (f) graph (g) the Y-intercept

1. $P(x) = (x + 1)(x)(x - 1)(x - 3)$
2. $P(x) = x(x - 1)(x - 2)(x - 3)$
3. $P(x) = x(x + 1)(x - 1)$
4. $P(x) = (x - 1)(x - 2)(x - 3)$
5. $P(x) = (x - 2)(x - 3)^2$
6. $P(x) = x^4 - 2x^2 - 8$

II. Without graphing determine: (a) the zeros; (b) x such that $P(x) > 0$; (c) x such that $P(x) < 0$

1. $P(x) = x^4 - 16$
2. $P(x) = x(x - 5)(x + 2)$
3. $P(x) = x^2 - 6x + 9$
4. $P(x) = x^2 - 7x + 12$
5. $P(x) = x^3 - 5x^2 + 6X$
6. $P(x) = x - 7$

SECTION IV EVALUATION

1. Review the Behavioral Objectives.
2. Take the Trial Run
3. Take the Test

ANSWERS:

EXERCISE 1

A. 1. 3; 2. 6; 3. 30; 4. 6;

B. 1. 3; 2. 4; 3. 5; 4. 0; 5. no degree

C. 1. $P(x) = a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0$

2. $P(x) = a_6x^6 + a_5x^5 + a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0$ 3. $P(x) = a_1x$

4. $P(x) = a_3x^3 + a_2x^2 + a_1x + a_0$ 5. $P(x) = a_2x^2 + a_1x + a_0$

D. 3 and 4

EXERCISE 2

A. 1. 7; 2. -1; B. 1. 27; 2. -15; C. 1. 58; 2. -7.9257

D. 1. -58; 2. -2

EXERCISE 3

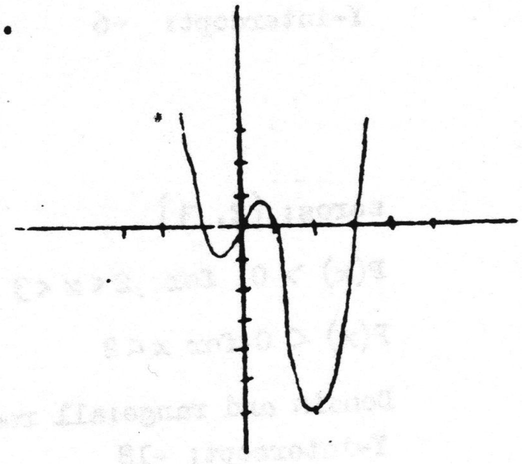
1. $\{-4, -1, -5\}$ 2. $\{1, 2, 9\}$ 3. $\{2, 3\}$ 4. $\{1, 2, 3\}$

5. $\left\{5, \frac{-5 \pm \sqrt{33}}{2}\right\}$ 6. $\{1, 2, 3, 4\}$ 7. $\{2, -1, 3\}$ 8. $\{1, 2, 3\}$

9. $\{-1, -2, -3\}$ 10. $\{1, -3, -5\}$ 11. $\{-2, 2\}$ 12. $\{0, 2\}$

EXERCISE 4

I. 1.



Zeros: $\{0, -1, 1, 3\}$

$P(x) > 0$ for $x < -1$ or $0 < x < 1$ or $x > 3$

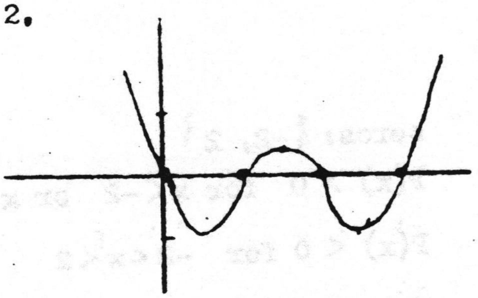
$P(x) < 0$ for $-1 < x < 0$ or $1 < x < 3$

Domain: All reals

Range: $\{y: y \geq -6\}$

Y-intercept: 0

2.



Zeros: $\{0, 1, 2, 3\}$

$P(x) > 0$ for $x < 0$ or $1 < x < 2$ or $x > 3$

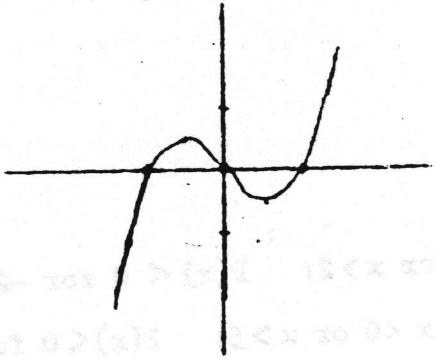
$P(x) < 0$ for $0 < x < 1$ or $2 < x < 3$

Domain: All reals

Range: $\left\{y: y \geq \frac{-15}{16}\right\}$

Y-intercept: 0

3.



Zeros: $\{-1, 0, 1\}$

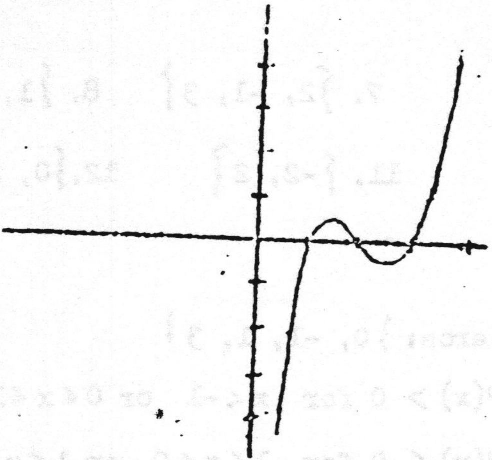
$P(x) > 0$ for $-1 < x < 0$ or $x > 1$

$P(x) < 0$ for $x < -1$ or $0 < x < 1$

Domain and Range: all reals

Y-intercept: 0

4.



zeros: {1, 2, 3}

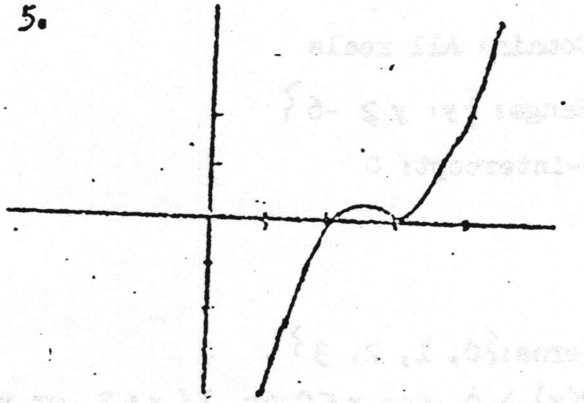
$P(x) > 0$ for $1 < x < 2$ or $x > 3$

$P(x) < 0$ for $x < 1$ or $2 < x < 3$

Domain and range: all reals

Y-intercept: -6

5.



zeros: {2, 3}

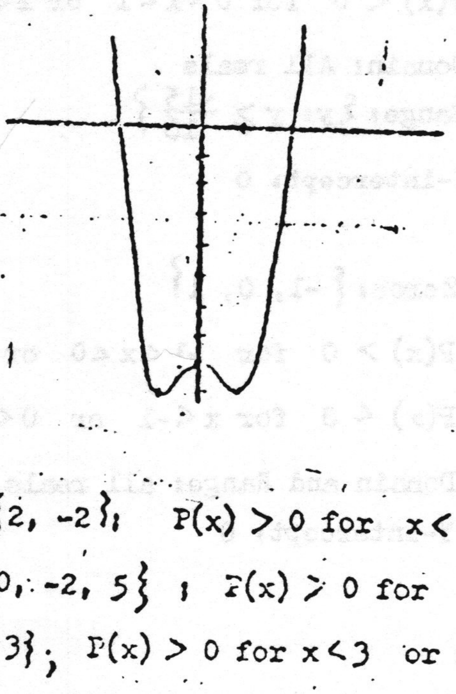
$P(x) > 0$ for $2 < x < 3$ or $x > 3$

$P(x) < 0$ for $x < 2$

Domain and range: all reals

Y-intercept: -18

6.



zeros: {-2, 2}

$P(x) > 0$ for $x < -2$ or $x > 2$

$P(x) < 0$ for $-2 < x < 2$

Y-intercept: -8

II.

1. Zeros: {2, -2}; $P(x) > 0$ for $x < -2$ or $x > 2$; $P(x) < 0$ for $-2 < x < 2$
2. Zeros: {0, -2, 5} ; $P(x) > 0$ for $-2 < x < 0$ or $x > 5$; $P(x) < 0$ for $x < -2$ or $0 < x < 5$
3. Zeros: {3}; $P(x) > 0$ for $x < 3$ or $x > 3$; $P(x) < 0$ for no x.
4. [illegible]s: {3, 4}; $P(x) > 0$ for $x < 3$ or $x > 4$; $P(x) < 0$ for $3 < x < 4$
5. Zeros: {0, 3, 2]; $P(x) > 0$ for $0 < x < 2$ or $x > 3$; $P(x) < 0$ for $x < 0$ or $2 < x < 3$
6. Zeros: {7}; $P(x) > 0$ for $x > 7$; $P(x) < 0$ for $x < 7$

ALGEBRA II

POLYNOMIAL FUNCTIONS
TRIAL RUN

I. Define

1. The Factor Theorem
2. The Zero Of A Function

II. Evaluate the following polynomials:

1. $P(x) = x^5 - 4x^3 + 5x - 2 \Longrightarrow P(2) =$

2. $P(x) = x^3 - 5x^2 + 8 \Longrightarrow P(-2) =$

3. $P(x) = x^4 - 5x^3 + 8x - 2 \Longrightarrow P(-1) =$

III. Factor and find the zeros of the following polynomial functions:

1. $x^4 + x^3 - x^2 - 7x - 6$
2. $x^3 - 39x + 70$
3. $x^3 - 9x^2 - 4x + 96$
4. $x^4 - 2x^3 - 4x^2 + 23x - 30$
5. $4x^3 + x^2 - 16x - 4$

IV. For each of the following determine:

a) the zeros, b) where $P(x) > 0$, c) where $P(x) < 0$,

d) the domain, e) the Range, f) and the graph

1. $P(x) = (x - 4)(x + 1)(x - 5)$
2. $P(x) = (x + 5)(x - 2)(x - 1)(x + 2)$
3. $P(x) = (x + 1)(x - 3)(x - 2)$
4. $P(x) = (x + 7)(x - 2)(x + 1)$
5. $P(x) = (x^2 - 4)(x^2 - 1)$

TRIAL RUN ANSWERS POLYNOMIALS FUNCTIONS

I. 1. The factor theorem: The polynomial p is divisible by $(x - a)$ if and only if $P(a) = 0$.

2. The zeroes of a function are those members "x" in the domain of f for which $f(x) = 0$.

II. 1. 8
2. -20
3. -4

III. 1. $\{-1, 2\}$

2. $\{2, 5, -7\}$

3. $\{-3, 4, 8\}$

4. $\{2, -3\}$

5. $\{-2, 2, -\frac{1}{4}\}$

IV. 1.
a) 4, -1, 5
b) $P(x) > 0 \Rightarrow -1 < x < 4, \quad x > 5$
c) $P(x) < 0 \Rightarrow x < -1, \quad +4 < x < 5$
d) D; Re
e) R: Re

2.
a) -5, 2, 1, -2
b) $P(x) > 0 \Rightarrow x < -5, \quad -2 < x < 1, \quad x > 2$
c) $P(x) < 0 \Rightarrow -5 < x < -2, \quad 1 < x < 2$
d) D: Re
e) R: $\left[\frac{-891}{16}, \infty\right)$

3.
a) -1, 3, 2
b) $P(x) > 0 \Rightarrow -1 < x < 2, \quad x > 3$
c) $P(x) < 0 \Rightarrow x < -1, \quad 2 < x < 3$
d) D: RE
e) R: Re

4.
a) -7, -1, 2
b) $P(x) > 0 \Rightarrow -7 < x < -1, \quad x > 2$
c) $P(x) < 0 \Rightarrow x < -7, \quad -1 < x < 2$
d) D: Re
e) R: Re

5.
a) 2, -2, 1, -1
b) $P(x) > 0 \Rightarrow x < -2, \quad -1 < x < 1, \quad x > 2$
c) $P(x) < 0 \Rightarrow -2 < x < -1, \quad 1 < x < 2$
d) D: Re
e) R: $\left[\frac{-35}{16}, \infty\right)$

[This is an approximation]